

Symmetry Reduction of the Two-Dimensional Ricci Flow Equation

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Abstract

This paper is devoted to obtain the one-dimensional group invariant solutions of the two-dimensional Ricci flow ((2D) Rf) equation. By classifying the orbits of the adjoint representation, the optimal system of one-dimensional subalgebras of the ((2D) Rf) equation is obtained. For each class, we will find the reduced equation by method of similarity reduction. By solving these reduced equations we will obtain new sets of group invariant solutions for the ((2D) Rf) equation.

Keywords: Lie symmetry group, two-dimensional Ricci flow equation, optimal system, group invariant solution

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1 Introduction

The Ricci flow was introduced by Hamilton in his seminal paper, “Three-manifolds with positive Ricci curvature” in 1982 [7]. Since then, Ricci flow has been a very useful tool for studying the special geometries which a manifold admits. Ricci flow is an evolution equation for a Riemannian metric which sometimes can be used in order to deform an arbitrary metric into a suitable metric that can specify the topology of the underlying manifold. If $(M, g(t))$ be a smooth Riemannian manifold, Ricci flow is defined by the equation

$$\frac{\partial}{\partial t} g(t) = -2Ric, \quad (1)$$

where Ric denotes the Ricci tensor of the metric g . By using the concept of Ricci flow, Grisha Perelman completely proved the Poincaré conjecture around 2003 [12-14]. The Ricci flow also is used as an approximation to the renormalization group flow for the two-dimensional nonlinear σ -model, in quantum field theory. see [6] and references therein.

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In this paper we want to obtain new solutions of the $((2D) \text{ Rf})$ equation by method of Lie symmetry group. As it is well known, the Lie symmetry group method has an important role in the analysis of differential equations. The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [8]. By this method we can reduce the order of ODEs and investigate the invariant solutions. Also we can construct new solutions from known ones (for more details about the applications of Lie symmetries, see [10, 1, 3]). Lei's method led to an algorithmic approach to find special solution of differential equation by its symmetry group. These solutions are called group invariant solutions and obtain by solving the reduced system of differential equation having fewer independent variables than the original system. Bluman and Cole generalized the Lie's symmetry method for finding the group-invariant solutions [2]. In this paper we apply this method to obtain the invariant solutions of $((2D) \text{ Rf})$ equation and classify them.

This paper is organized as follows. In section 2, by using the mechanical model of Ricci flow, Lie symmetries of $((2D) \text{ Rf})$ equation will be state and some results yield from the structure of the Lie algebra of the Lie symmetry group. In section 3, we will construct an optimal system of one-dimensional subalgebras of the $((2D) \text{ Rf})$ equation which is useful for classifying of group invariant solutions. In section 4, the reduced equation for each element of optimal system is obtained. In section 5, we will solve the reduced equations by method of Lie symmetry group and obtain the group invariant solutions of $((2D) \text{ Rf})$ equation.

2 Lie Symmetries of $((2D) \text{ Rf})$ Equation

As we know, transformations which map solutions of a differential equation to other solutions are called symmetries of the equation. In [10] Olver has introduced the procedure of finding Lie symmetry group of a differential equation. Cimpoeas and Constantinescu have introduced the mechanical model for the Ricci flow as follows:

$$u^2 u_t + u_y u_x - u u_{xy} = 0 \quad (2)$$

and obtained the Lie symmetry group of this equation [5]. They proved that this equation admits a 6-parameter Lie group, G , with the following infinitesimal generators for its Lie algebra, \mathfrak{g} .

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_y, & X_3 &= \partial_t, \\ X_4 &= t\partial_t + u\partial_u, & X_5 &= x\partial_x - u\partial_u, & X_6 &= y\partial_y - u\partial_u. \end{aligned} \quad (3)$$

The commutator table of Lie algebra for \mathfrak{g} is given below, where the entry in the i^{th} row and j^{th} column is $[X_i, X_j] = X_i X_j - X_j X_i$, $i, j = 1, \dots, 6$.

Exponentiating the infinitesimal symmetries (3) we obtain the one-parameter groups $g_k(s)$ generated by X_k , $k = 1, \dots, 6$:

$$\begin{aligned} g_1(s) &: (x, y, t, u) \longmapsto (x + s, y, t, u), \\ g_2(s) &: (x, y, t, u) \longmapsto (x, y + s, t, u), \\ g_3(s) &: (x, y, t, u) \longmapsto (x, y, t + s, u), \\ g_4(s) &: (x, y, t, u) \longmapsto (x, y, te^s, ue^s), \\ g_5(s) &: (x, y, t, u) \longmapsto (xe^s, y, t, ue^{-s}), \\ g_6(s) &: (x, y, t, u) \longmapsto (x, ye^s, t, ue^{-s}), \end{aligned} \quad (4)$$

Table 1: The commutator table of \mathfrak{g} .

$[\cdot, \cdot]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	X_1	0
X_2	0	0	0	0	0	X_2
X_3	0	0	0	X_3	0	0
X_4	0	0	$-X_3$	0	0	0
X_5	$-X_1$	0	0	0	0	0
X_6	0	$-X_2$	0	0	0	0

Consequently, we can state the following theorem:

Theorem 2.1. *If $f = f(x, y, t)$ is a solution of (2), so are functions*

$$\begin{aligned}
 g_1(s).f &= f(x - s, y, t), & g_4(s).f &= f(x, y, te^{-s})e^s, \\
 g_2(s).f &= f(x, y - s, t), & g_5(s).f &= f(xe^{-s}, y, t)e^{-s}, \\
 g_3(s).f &= f(x, y, t - s), & g_6(s).f &= f(x, ye^{-s}, t)e^{-s}.
 \end{aligned} \tag{5}$$

3 One-dimensional optimal system of subalgebras for the ((2D) Rf) equation

In this section, we obtain the one-dimensional optimal system of ((2D) Rf) equation by using symmetry group. Since every linear combination of infinitesimal symmetries is an infinitesimal symmetry, there is an infinite number of one-dimensional subgroups for G . Therefore, it's important to determine which subgroups give different types of solutions. For this, we must find invariant solutions which can not be transformed to each other by symmetry transformations in the full symmetry group. This led to concept of an optimal system of subalgebra. For one-dimensional subalgebras, this classification problem is the same as the problem of classifying the orbits of the adjoint representation [10]. Optimal set of subalgebras is obtaining from taking only one representative from each class of equivalent subalgebras. The problem of classifying the orbits is solved by taking a general element in the lie algebra and simplify it as much as possible by imposing various adjoint transformation on it [11, 9]. Adjoint representation of each X_i , $i = 1, \dots, 6$ is defined as follow:

$$\text{Ad}(\exp(s.X_i).X_j) = X_j - s.[X_i, X_j] + \frac{s^2}{2}.[X_i, [X_i, X_j]] - \dots, \tag{6}$$

where s is a parameter and $[X_i, X_j]$ is the commutator of the Lie algebra for $i, j = 1, \dots, 6$ ([10], page 199). Taking into account the table of commutator, we can compute all the adjoint representations corresponding to the Lie group of the ((2D) Rf) equation. They are presented in Table 2. Not that, the (i, j) entry indicate $\text{Ad}(\exp(s.X_i).X_j)$.

Now we can state the following theorem:

Theorem 3.1. *A one-dimensional optimal system for Lie algebra of ((2D) Rf) equation is given by*

$$\begin{aligned}
 1) X_4 + aX_5 + bX_6, & \quad 4) \varepsilon X_1 + \varepsilon' X_2 + X_4, & 7) \varepsilon X_1 + \varepsilon' X_3 + X_6, \\
 2) \varepsilon X_2 + X_4 + aX_5, & 5) \varepsilon X_3 + X_5 + aX_6, & 8) \varepsilon X_1 + cX_2 + \varepsilon' X_3, \\
 3) \varepsilon X_1 + X_4 + aX_6, & 6) \varepsilon X_2 + \varepsilon' X_3 + X_5, &
 \end{aligned} \tag{7}$$

Table 2: The adjoint representation table of the infinitesimal generators X_i .

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	X_4	$X_5 - sX_1$	X_6
X_2	X_1	X_2	X_3	X_4	X_5	$X_6 - sX_2$
X_3	X_1	X_2	X_3	$X_4 - sX_3$	X_5	X_6
X_4	X_1	X_2	$e^s X_3$	X_4	X_5	X_6
X_5	$e^s X_1$	X_2	X_3	X_4	X_5	X_6
X_6	X_1	$e^s X_2$	X_3	X_4	X_5	X_6

where ε and ε' are ± 1 or zero. Also $a, b, c \in \mathbb{R}$ and $a \neq 0, b \neq 0$.

Proof. Let $F_i^s : \mathfrak{g} \rightarrow \mathfrak{g}$ be the adjoint transformation defined by $X \mapsto \text{Ad}(\exp(sX_i)).X$, for $i = 1, \dots, 6$. The matrix of F_i^s , $i = 1, \dots, 6$, with respect to basis $\{X_1, \dots, X_6\}$ is:

$$M_1^s = \begin{bmatrix} 1 & 0 & 0 & 0 & -s & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -s \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_3^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_4^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_5^s = \begin{bmatrix} e^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_6^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^s & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

respectively. If $X = \sum_{i=1}^6 a_i X_i$, then we have

$$F_6^{s_6} \circ F_5^{s_5} \circ \dots \circ F_1^{s_1} : X \mapsto e^{s_5} a_1 X_1 + e^{s_6} a_2 X_2 + e^{s_4} a_3 X_3 + (a_4 - e^{s_4} s_3 a_3) X_4 + (a_5 - e^{s_5} s_1 a_1) X_5 + (a_6 - e^{s_6} s_2 a_2) X_6.$$

Now, we try to vanish the coefficients of X by acting the adjoint representations $M_i^{s_i}$ on X , by choosing suitable parameters s_i in each step. Therefor we can simplify X as follows:

If $a_4 \neq 0$, $a_5 \neq 0$ and $a_6 \neq 0$, we can make the coefficients of X_3 , X_1 and X_2 vanish by $F_3^{s_3}$, $F_1^{s_1}$ and $F_2^{s_2}$. By setting $s_3 = -\frac{a_3}{a_4}$, $s_1 = -\frac{a_1}{a_5}$ and $s_2 = -\frac{a_2}{a_6}$ respectively. Scaling X if necessary, we can assume that $a_4 = 1$. So X is reduced to the case (1).

If $a_4 \neq 0$, $a_5 \neq 0$ and $a_6 = 0$, we can make the coefficients of X_3 and X_1 vanish by $F_3^{s_3}$ and $F_1^{s_1}$; By setting $s_3 = -\frac{a_3}{a_4}$ and $s_1 = -\frac{a_1}{a_5}$, respectively. Also by setting $s_6 = \ln|a_2|$ in $F_6^{s_6}$, we can make the coefficient of X_2 vanish or ± 1 . Scaling X if necessary, we can assume that $a_4 = 1$. So X is reduced to the case (2).

If $a_4 \neq 0$, $a_6 \neq 0$ and $a_5 = 0$, we can make the coefficients of X_3 and X_2 vanish by $F_3^{s_3}$ and $F_2^{s_2}$; By setting $s_3 = -\frac{a_3}{a_4}$ and $s_2 = -\frac{a_2}{a_6}$, respectively. Also by setting $s_5 = \ln|a_1|$ in $F_5^{s_5}$, we can make the coefficient of X_1 vanish or ± 1 . Scaling X if necessary, we can assume that $a_4 = 1$. So X is reduced to the case (3).

If $a_4 \neq 0$ and $a_5 = a_6 = 0$, we can make the coefficient of X_3 vanish by $F_3^{S_3}$; By setting $s_3 = -\frac{a_3}{a_4}$. Also by setting $s_5 = \ln|a_1|$ and $s_6 = \ln|a_2|$ in $F_5^{S_5}$ and $F_6^{S_6}$, we can make the coefficients of X_1 and X_2 vanish or ± 1 , respectively. Scaling X if necessary, we can assume that $a_4 = 1$. So X is reduced to the case (4).

If $a_5 \neq 0$, $a_6 \neq 0$ and $a_4 = 0$, we can make the coefficients of X_1 and X_2 vanish by $F_1^{S_1}$ and $F_2^{S_2}$; By setting $s_1 = -\frac{a_1}{a_5}$ and $s_2 = -\frac{a_2}{a_6}$, respectively. Also by setting $s_4 = \ln|a_3|$ in $F_4^{S_4}$, we can make the coefficient of X_1 vanish or ± 1 . Scaling X if necessary, we can assume that $a_5 = 1$. So X is reduced to the case (5).

If $a_5 \neq 0$ and $a_4 = a_6 = 0$, we can make the coefficient of X_1 vanish by $F_1^{S_1}$; By setting $s_1 = -\frac{a_1}{a_5}$. Also by setting $s_6 = \ln|a_2|$ and $s_4 = \ln|a_3|$ in $F_6^{S_6}$ and $F_4^{S_4}$, we can make the coefficients of X_2 and X_3 vanish or ± 1 , respectively. Scaling X if necessary, we can assume that $a_5 = 1$. So X is reduced to the case (6).

If $a_4 = a_5 = 0$ and $a_6 \neq 0$, we can make the coefficient of X_2 vanish by $F_2^{S_2}$; By setting $s_2 = -\frac{a_2}{a_6}$. Also by setting $s_5 = \ln|a_1|$ and $s_4 = \ln|a_3|$ in $F_5^{S_5}$ and $F_4^{S_4}$, we can make the coefficients of X_1 and X_3 vanish or ± 1 , respectively. Scaling X if necessary, we can assume that $a_6 = 1$. So X is reduced to the case (7).

If $a_4 = a_5 = a_6 = 0$, we can make the coefficients of X_1 and X_3 vanish or ± 1 by $F_5^{S_5}$ and $F_4^{S_4}$; By setting $s_5 = \ln|a_1|$ and $s_4 = \ln|a_3|$, respectively. So X is reduced to the case (8).

□

4 Similarity Reduction of ((2D) Rf) Equation

In this section, the two-dimensional Ricci flow equation will be reduced by expressing it in the new coordinates. The ((2D) Rf) equation is expressed in the coordinates (x, y, t, u) , we must search for this equation's form in the suitable coordinates for reducing it. These new coordinates will be obtained by looking for independent invariants (z, w, f) corresponding to the generators of the symmetry group. Hence, By using the new coordinates and applying the chain rule, we obtain the reduced equation. We express this procedure for one of the infinitesimal generators in the optimal system (7) and list the result for some other cases.

For example, consider the case (7) in theorem (3.1) when $\varepsilon = 1$ and $\varepsilon' = 0$, therefor we have $X := X_1 + X_6$. For determining independent invariants I , we ought to solve the PDEs $X(I) = 0$, that is

$$(X_1 + X_6)I = (\partial_x + y\partial_y - u\partial_u)I = \frac{\partial I}{\partial x} + y\frac{\partial I}{\partial y} + 0\frac{\partial I}{\partial t} - u\frac{\partial I}{\partial u} = 0. \quad (8)$$

For solving this PDE, the following associated characteristic ODE must be solved:

$$\frac{dx}{1} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{-u}. \quad (9)$$

Hence, three functionally independent invariants $z = ye^{-x}$, $w = t$ and $f = uy$ are obtained. If we treat f as a function of z and w , we can compute formulae for the derivatives of u

with respect to x , y and t in terms of z , w , f and the derivatives of f with respect to z and w . By using the chain rule and the fact that $u = f(z, w)y^{-1}$ we have

$$\begin{aligned} u_t &= (f_z z_t + f_w w_t)y^{-1} = f_w y^{-1}, & u_x &= -f_z e^{-x}, \\ u_y &= f_z e^{-x} y^{-1} - f y^{-2}, & u_{xy} &= -e^{-2x} f_{zz}. \end{aligned} \quad (10)$$

After substituting the above relations into the equation (2), we obtain:

$$u^2 u_t + u_y u_x - u u_{xy} = y^{-3} (f^2 f_w - f_z^2 z^2 + f f_z z + f f_{zz} z^2) = 0. \quad (11)$$

So the reduced equation is

$$f^2 f_w - z^2 f_z^2 + z f f_z + z^2 f f_{zz} = 0. \quad (12)$$

This equation has two independent variables z and w and one dependent variable f . In a similar way, we can compute all of the similarity reduction equations corresponding to the infinitesimal symmetries that mentioned in theorem (3.1). some of them are listed in Table 3.

Table 3: Lie invariants, similarity solutions and reduced equations.

i	\mathfrak{h}_i	$\{z_i, w_i, v_i\}$	u_i	Similarity reduced equations
1	$X_1 + X_6$	$\{y e^{-x}, t, u y\}$	$\frac{f(z, w)}{y}$	$f^2 f_w - z^2 f_z^2 + z f f_z + z^2 f f_{zz} = 0$
2	$X_2 + X_4$	$\{x, t e^{-y}, u e^{-y}\}$	$f(z, w) e^y$	$f^2 f_w - w f_z f_w + w f f_{zw} = 0$
3	$X_3 + X_5 + a X_6$	$\{\frac{y}{x^a}, \ln \frac{e^t}{x}, u x^{a+1}\}$	$\frac{f(z, w)}{x^{a+1}}$	$f_w (f^2 - f_z) - a z f_z^2 + f (a f_z + a z f_{zz} + f_{zw}) = 0$
4	$X_5 + a X_6$	$\{\frac{y}{x^a}, t, u x^{a+1}\}$	$\frac{f(z, w)}{x^{a+1}}$	$f^2 f_w + a (f f_z - z f_z^2 + z f f_{zz}) = 0$
5	$X_2 + X_3 + X_5$	$\{\ln \frac{e^y}{x}, \ln \frac{e^t}{x}, u x\}$	$\frac{f(z, w)}{x}$	$f^2 f_w - f_z^2 - f_w f_z + f f_{zz} + f f_{zw} = 0$
6	$X_2 + X_5$	$\{\ln \frac{e^y}{x}, t, u x\}$	$\frac{f(z, w)}{x}$	$f^2 f_w - f_z^2 + f f_{zz} = 0$
7	$-X_3 + X_6$	$\{x, t + \ln y, u y\}$	$\frac{f(z, w)}{y}$	$f^2 f_w + f_z f_w - f f_{zw} = 0$
8	$X_1 + a X_2$	$\{y - a x, t, u\}$	$f(z, w)$	$f^2 f_w - a f_z^2 + a f f_{zz} = 0$
9	$a X_2 + X_3$	$\{x, \frac{t a - y}{a}, u\}$	$f(z, w)$	$a f^2 f_w - f_z f_w + f f_{zw} = 0$

5 Group Invariant Solutions of ((2D) Rf) Equation

In this section we reduce the equations obtained in last section to ODEs and solve them.

For example, the equation (12) admits a 4-parameters family of Lie operators with following infinitesimal generators

$$\begin{aligned} V_1 &= \frac{1}{2} z \ln z \partial_z + w \partial_w, & V_3 &= -\frac{1}{2} z \ln z \partial_z + f \partial_f, \\ V_2 &= \partial_w, & V_4 &= z \partial_z. \end{aligned} \quad (13)$$

The invariants associated to the infinitesimal generator V_2 , are $s = z$ and $g = f$. By substituting these invariants into the equation (12) and using chain rule, the reduced equation is obtained as follows:

$$s g'^2 - g g' - s g g'' = 0 \quad (14)$$

the solution of this equation is $g(s) = c_2 s^{c_1} = c_2 z^{c_1}$. therefore we have $f(z) = c_2 z^{c_1} = c_2 (ye^{-x})^{c_1} = c_2 y^{c_1} e^{-c_1 x}$. So $u = fy^{-1} = c_2 y^{c_1-1} e^{-c_1 x}$ is a solution of ((2D) Rf) equation.

By similar arguments, we can obtain other invariant solutions of the equation (12). Also by reducing other equations in Table 3, we can find other solutions of ((2D) Rf) equation. Some of the similarity reduced equations and their invariant solutions are listed in Table 4 and Table 5 respectively.

Conclusion

In this paper, by using the adjoint representation of the symmetry group on its Lie algebra, we have constructed an optimal system of one-dimensional subalgebras of the two-dimensional Ricci flow equation. Moreover, we have obtained the similarity reduced equations for each element of optimal system as well as some group invariant solutions of two-dimensional Ricci flow equation.

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Table 4: ODEs that obtain from the reduced equations of table 3.

i	Simmetry group generators	Optimal system	Invariants $\{s, g\}$	Reduced equation
1	$V_1 = \frac{1}{2}z\ln z\partial_z + w\partial_w$	$\mathcal{A}_1^1 : V_2$	$\{z, f\}$	$sg'^2 - gg' - sgg'' = 0$
	$V_2 = \partial_w$	$\mathcal{A}_1^2 : V_3$	$\{w, f(\ln z)^2\}$	$g^2(g' + 2) = 0$
	$V_3 = -\frac{1}{2}z\ln z\partial_z + f\partial_f$	$\mathcal{A}_1^3 : V_1 + V_3$	$\{z, \frac{f}{w}\}$	$g^3 - s^2g'^2 + sgg' + s^2gg'' = 0$
	$V_4 = z\partial_z$	$\mathcal{A}_1^4 : V_2 + V_4$	$\{w - \ln z, f\}$	$g^2g' - g'^2 + gg'' = 0$
2	$V_1 = w\partial_w + f\partial_f$	$\mathcal{A}_2^1 : V_2$	$\{w, fz\}$	$g^2g' = 0$
	$V_2 = z\partial_z - f\partial_f$	$\mathcal{A}_2^2 : V_3$	$\{w, f\}$	$g^2g' = 0$
	$V_3 = \partial_z$			
3	$V_1 = z\ln z\partial_z + w\partial_w - f(1 + \ln z)\partial_f$	$\mathcal{A}_3^1 : V_1$	$\{\frac{w}{\ln z}, fz\ln z\}$	$gg'(g + 2as - 1) + ag^2 + s(as - 1)(gg'' - g'^2) = 0$
	$V_2 = \partial_w$	$\mathcal{A}_3^2 : V_2$	$\{z, f\}$	$sg'^2 - gg' - sgg'' = 0$
	$V_3 = z\partial_z - f\partial_f$	$\mathcal{A}_3^3 : V_3$	$\{w, fz\}$	$g^2g' = 0$
4	$V_1 = z\partial_z + w\partial_w$	$\mathcal{A}_4^1 : V_4$	$\{w, fz(4 - \ln z(4 - \ln z))\}$	$g^2(g' + 2a) = 0$
	$V_2 = \partial_w$			
	$V_3 = -z\partial_z + f\partial_f$	$\mathcal{A}_4^2 : V_1 + V_3$	$\{z, \frac{f}{w}\}$	$g^3 - a(sg'^2 + gg' + sgg'') = 0$
5	$V_1 = z\partial_z + w\partial_w - f\partial_f$	$\mathcal{A}_5^1 : V_1$	$\{\frac{w}{z}, fz\}$	$gg'(1 - g - 2s) - g^2 + s(s - 1)(g'^2 - gg'') = 0$
	$V_2 = \partial_w$	$\mathcal{A}_5^2 : V_2$	$\{z, f\}$	$-g'^2 + gg'' = 0$
	$V_3 = \partial_z$	$\mathcal{A}_5^3 : V_2 - V_3$	$\{z + w, f\}$	$g^2g' - 2g'^2 + 2sgg'' = 0$
6	$V_1 = \frac{z}{2}\partial_z + w\partial_w$	$\mathcal{A}_6^1 : V_3$	$\{w, fz^2\}$	$g' + 2 = 0$
	$V_2 = \partial_w$	$\mathcal{A}_6^2 : V_1 + V_3$	$\{z, \frac{f}{w}\}$	$g^3 - g'^2 + gg'' = 0$
	$V_3 = -\frac{z}{2}\partial_z + f\partial_f$	$\mathcal{A}_6^3 : V_2 + V_4$	$\{w - z, f\}$	$g^2g' - g'^2 + gg'' = 0$
	$V_4 = \partial_z$			
7	$V_1 = w\partial_w$	$\mathcal{A}_7^1 : V_4$	$\{w, f\}$	$g' = 0$
	$V_2 = \partial_w$	$\mathcal{A}_7^2 : V_2 + V_3$	$\{w - \ln z, zf\}$	$g^2g' - g'^2 + gg'' = 0$
	$V_3 = z\partial_z - f\partial_f$			
	$V_4 = \partial_z$			
8	$V_1 = \frac{z}{2}\partial_z + w\partial_w$	$\mathcal{A}_8^1 : V_2$	$\{z, f\}$	$g'^2 - gg'' = 0$
	$V_2 = \partial_w$	$\mathcal{A}_8^2 : V_3$	$\{w, fz^2\}$	$g^2(g' + 2a) = 0$
	$V_3 = -\frac{z}{2}\partial_z + f\partial_f$	$\mathcal{A}_8^3 : V_2 + V_4$	$\{w - z, f\}$	$g^2g' - ag'^2 + agg'' = 0$
	$V_4 = \partial_z$	$\mathcal{A}_8^4 : V_1 - V_3$	$\{\frac{w}{z}, fz\}$	$g^2g' - as^2g'^2 + 2asgg' + ag^2 + as^2gg'' = 0$
9	$V_1 = w\partial_w$			
	$V_3 = z\partial_z - f\partial_f$	$\mathcal{A}_9^1 : V_2 + V_4$	$\{w - z, f\}$	$ag^2g' + g'^2 - gg'' = 0$
	$V_2 = \partial_w$	$\mathcal{A}_9^2 : V_2 + V_3$	$\{w - \ln z, zf\}$	$ag^2g' + g'^2 - gg'' = 0$
	$V_4 = \partial_z$			

Table 5: Group invariant solutions of the ((2D) Rf) equation.

\mathcal{A}_i^j	Invariant solution	Solution after substituting
\mathcal{A}_1^1	$c_2 s^{c_1}$	$c_2 y^{c_1-1} e^{-c_1 x}$
\mathcal{A}_1^2	$-2s + c_1$	$\frac{-2t+c_1}{y(\ln y-x)^2}$
\mathcal{A}_1^3	$\frac{1}{2c_1^2}(1 - \tanh(\frac{\ln s-c_2}{2c_1})^2)$	$\frac{t}{2c_1^2 y}(1 - \tanh(\frac{\ln y-x-c_2}{2c_1})^2)$
\mathcal{A}_1^4	$\frac{c_1 e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$	$\frac{c_1 e^{c_1(x+t-\ln y+c_2)}}{-y+y e^{c_1(x+t-\ln y+c_2)}}$
\mathcal{A}_2^1	c_1	$\frac{c_1}{x} e^y$
\mathcal{A}_2^2	c_1	$c_1 e^y$
\mathcal{A}_3^1	$\frac{c_1(1+c_1)s^{c_1}}{-s^{c_1}(1+c_1-as)+ac_1c_2(1+c_1)(as-1)^{c_1+1}}$	$\frac{c_1(1+c_1)(\frac{t-\ln x}{\ln y-\ln x})^{c_1}}{xy \ln \frac{y}{x} - (\frac{t-\ln x}{\ln y-\ln x})^{c_1}(1+c_1 - \frac{at-a\ln x}{\ln y-\ln x}) + ac_1c_2(1+c_1)(\frac{at-a\ln x}{\ln y-\ln x}-1)^{c_1+1}}$
\mathcal{A}_3^2	$c_2 s^{c_1}$	$\frac{c_2}{x^2} (\frac{y}{x})^{c_1}$
\mathcal{A}_3^3	c_1	$\frac{c_1}{yx}$
\mathcal{A}_4^1	$-2as + c_1$	$\frac{-2at+c_1}{xy(4-4\ln \frac{y}{x} + (\ln \frac{y}{x})^2)}$
\mathcal{A}_4^2	$\frac{1}{2c_1 s}(1 - \tanh(\frac{\ln s+c_2}{2\sqrt{ac_1}})^2)$	$\frac{t}{2c_1 xy}(1 - \tanh(\frac{\ln y-\ln x+c_2}{2\sqrt{ac_1}})^2)$
\mathcal{A}_5^1	$\frac{c_1(1+c_1)s^{c_1}}{-s^{c_1}(1+c_1-s)+c_1c_2(1+c_1)(s-1)^{c_1+1}}$	$\frac{c_1(1+c_1)(\frac{t-\ln x}{y-\ln x})^{c_1}}{x(y-\ln x) - (\frac{t-\ln x}{y-\ln x})^{c_1}(1+c_1 - \frac{t-\ln x}{y-\ln x}) + c_1c_2(1+c_1)(\frac{t-y}{y-\ln x})^{c_1+1}}$
\mathcal{A}_5^2	$c_2 e^{c_1 s}$	$\frac{c_2}{x} e^{c_1(y-\ln x)}$
\mathcal{A}_5^3	$\frac{2c_1 e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$	$\frac{2c_1 e^{c_1(t+y-2\ln x+c_2)}}{-x+x e^{c_1(t+y-2\ln x+c_2)}}$
\mathcal{A}_6^1	$-2s + c_1$	$\frac{-2t+c_1}{x(y-\ln x)^2}$
\mathcal{A}_6^2	$\frac{1}{2c_1^2}(1 - \tanh(\frac{s+c_2}{2c_1})^2)$	$\frac{t}{2c_1^2 x}(1 - \tanh(\frac{y-\ln x+c_2}{2c_1})^2)$
\mathcal{A}_6^3	$\frac{c_1 e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$	$\frac{c_1 e^{c_1(t-y+\ln x+c_2)}}{-x+x e^{c_1(t-y+\ln x+c_2)}}$
\mathcal{A}_7^1	c_1	$\frac{c_1}{y}$
\mathcal{A}_7^2	$\frac{c_1 e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$	$\frac{c_1 e^{c_1(\ln y-\ln x+t+c_2)}}{xy(-1+e^{c_1(\ln y-\ln x+t+c_2)})}$
\mathcal{A}_8^1	$c_2 e^{c_1 s}$	$c_2 e^{c_1(y-ax)}$
\mathcal{A}_8^2	$-2as + c_1$	$\frac{-2at+c_1}{(y-ax)^2}$
\mathcal{A}_8^3	$\frac{c_1 a e^{c_1(s+c_2)}}{-1+e^{c_1(s+c_2)}}$	$\frac{c_1 a e^{c_1(ax-y+t+c_2)}}{-1+e^{c_1(ax-y+t+c_2)}}$
\mathcal{A}_8^4	$\frac{-c_1^2 a e^{\frac{c_1}{s}}}{e^{\frac{c_1}{s}}(c_1-s)-sac_1^2c_2}$	$\frac{-c_1^2 a e^{\frac{c_1(y-ax)}{t}}}{e^{\frac{c_1(y-ax)}{t}}(c_1 y - c_1 a x - t) - c_1^2 c_2 a t}$
\mathcal{A}_9^1	$\frac{c_1 e^{c_1(s+c_2)}}{1-ae^{c_1(s+c_2)}}$	$\frac{c_1 e^{c_1(t-x-\frac{y}{a}+c_2)}}{1-ae^{c_1(t-x-\frac{y}{a}+c_2)}}$
\mathcal{A}_9^2	$\frac{c_1 e^{c_1(s+c_2)}}{1-ae^{c_1(s+c_2)}}$	$\frac{g_1 e^{c_1(t-\ln x-\frac{y}{a}+c_2)}}{c_1(t-\ln x-\frac{y}{a}+c_2)}$